|||| 1.0

||| 1.1

|||| 1.25 |||| 1.4 ||| 1.6

2.8   2.5

3.2   2.2

3.6

4.0   2.0

1.8

MICROCOPY RESOLUTION TEST CHART

②

AFATL-TR-86-90

Boundary Element (Integral) Solutions to Heat Conduction Problems

Jim Sirkis

CLUSTERS AND WARHEADS BRANCH
MUNITIONS DIVISION

DECEMBER 1986

FINAL REPORT FOR PERIOD MAY 1986 – JULY 1986

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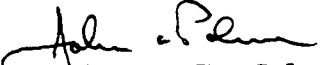
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FOR THE COMMANDER

JOHN A. PALMER, Colonel, USAF
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This study discusses the analytical and implementational difficulties associated with transient scalar boundary element methods. The fundamental solutions to steady and transient heat conduction problems are derived from first principles. Their implementation in the solution to seven representative heat conduction problems is detailed.

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PREFACE

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LIST OF SYMBOLS

L	Linear Differential Operator
f	boundary condition on u
g	boundary condition an $\frac{\partial u}{\partial n}$
Γ	boundary curve
Γ_1	part of the boundary curve on which f is prescribed
Γ_2	part of the boundary curve on which g is prescribed
B	domain of interest
G	fundamental solution
$d\ell$	differential arc length element
$d\sigma$	differential surface element
L*	adjoint of L
$\delta(\underset{\sim}{r}-\underset{\sim}{r}_0)$	Dirac delta function
ε	radius of a small semi-circle circumventing a singularity
$k_0(x)$	zero order modified Bessel function of the second kind
k	thermal conductivity
s	Laplace transform
T	Laplace transform of T

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SECTION I

INTRODUCTION

Boundary element method (B.E.M) solutions to seven representative two-dimensional heat conduction problems are presented. The steady heat conduction through an infinite rectangular prism is the first case discussed. Each of the next six cases present increasingly more complex governing equations and boundary conditions. The discussion culminates in the examination of heat conduction through an infinite circular prism with heat radiated from its surface. The fundamental solution is derived for each class of governing equations and its implementation is discussed for each specific case. The time dependent problems are successfully solved via the Laplace transform method. In all cases excellent results are achieved using very few grid points.

This report represents the preliminary research in using the B.E.M. to study the non-linear coupled partial differential equations associated with plastic wave propagation. Its intent is to expose the analytical and implementational difficulties associated with application of boundary element methods to transient problems.

SECTION II

THE BOUNDARY ELEMENT METHOD

A two-dimensional linear differential operator will be used to introduce the concepts behind the boundary element method. There is no loss of generality in specifying a two-dimensional operator since the three-dimensional development is identical.

$$L(u) = \phi \tag{1}$$

and
$$u = f \quad \text{on} \quad \Gamma_1 \tag{2a}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma_2 \tag{2b}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ is the bounding curve of surface B (Figure 1). If the idea that there is a fundamental solution associated Equation 1 is accepted, then the boundary element method can be developed following Rizzo (Reference 1), the solution to Equation 1 can be written as

$$u(p) = - \int_{\Gamma_2} u(Q)\frac{\partial G}{\partial n}(p,Q)d\ell + \int_{\Gamma_1} G(p,Q)\frac{\partial u}{\partial n}(Q)d\ell - \int_B \phi(p,Q)G(p,Q)d\sigma + W(p) \tag{3}$$

where

$$W(p) = - \int_{\Gamma_1} f(Q)\frac{\partial G}{\partial n}(p,Q)d\ell + \int_{\Gamma_2} g(Q)G(p,Q)d\ell.$$
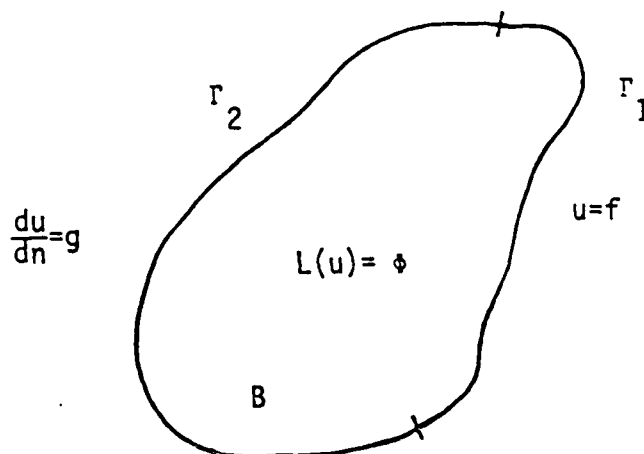

Figure 1. General Two-Dimensional Domain

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Here, p is any point in B, Q is any point on Γ, dℓ is a differential arc length element and dσ is a differential surface element. An integral equation for the unknown u(Q) and $\frac{\partial u}{\partial n}$(Q) can be obtained through the limiting process lim u(p) where P is any point on the bounding curve. This process yields
p→P

$$cu(P) = - \int_{\Gamma_2} u(Q)\frac{\partial G}{\partial n}(P,Q)d\ell + \int_{\Gamma_1} G(P,Q)\frac{\partial u}{\partial n}(Q)d\ell - \int_B \phi(P,Q)G(P,Q)d\sigma + W(P) \quad (4)$$

with

$$W(P) = - \int_{\Gamma_1} f(Q)\frac{\partial G}{\partial n}(P,Q)d\ell + \int_{\Gamma_2} g(Q)G(P,Q)d\ell.$$

where c depends on the surface roughness and the singularity contained in G. It is now possible to find the unknown functions u(Q) and $\frac{\partial u}{\partial n}$(Q). The substitution of these functions into Equation 3 gives the solution to L(u) = ϕ in B. Therefore, only knowledge of the function on the boundary is necessary to find the functional value at any interior point.

The numerical solution to Equation 4, and subsequently Equation 3, can be effected by first descretizing the boundary curve into N segments, i.e. Γ_i, i=1,2,3...N. Next, the functions u(Q) and $\frac{\partial u}{\partial n}$(Q) are approximated by suitable polynomials over each segment. In this report u(Q) and $\frac{\partial u}{\partial n}$(Q) are approximated by zero order polynomials, that is, u(Q) and $\frac{\partial u}{\partial n}$(Q) are assumed constant over each boundary segment. The development for higher order polynomials is similar and is given by Brebbia (Reference 2). Based on the aforementioned assumptions Equation 4 can be written as

$$-cu(P) - \sum_{i=1}^{N_1} u(Q_i) \int_{\Gamma_2}\frac{\partial G}{\partial n}(P,Q_i)d\ell_i + \sum_{i=1}^{N_2} \frac{\partial u}{\partial n}(Q_i) \int_{\Gamma_1} (P,Q_i)d\ell_i = Y(P) \quad (5)$$

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with

$$Y(P) = + \sum_{i=1}^{N} f(Q_i) \int_{\Gamma_1} \frac{\partial G}{\partial n}(P,Q_i)d\ell_i - \sum_{i=1}^{N} g(Q_i) \int_{\Gamma_2} G(P,Q_i)d\ell_i$$

$$+ \sum_{i=1}^{N} \int_B \phi(P,Q_i)G(P,Q_i)d\sigma_1$$

and, therefore can be reduced to

$$[A - IC]\{u\} + [M]\{\frac{du}{dn}\} = \{Y\} \tag{6}$$

where

$$\{u\} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ u_{N-2} \\ u_{N-1} \\ u_N \end{bmatrix} \qquad \{g\} = \begin{bmatrix} \frac{\partial u}{\partial n_1} \\ \frac{\partial u}{\partial n_2} \\ \frac{\partial u}{\partial n_3} \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_{ij} = \int_{\Gamma_1} \frac{\partial G}{\partial n}(P_j,Q_i)d\ell_i \quad \text{and} \quad M_{ij} = \int_{\Gamma_2} G(P_j,Q_i)d\ell_i;$$

where I is the identity matrix, N_1 and N_2 are the number of nodes where $\frac{\partial u}{\partial n}$ and u are unknown, respectively. Since $\{u\}$ has zero elements in the positions where $\{\frac{\partial u}{\partial n}\}$ has non-zero elements and vice versa, Equation 6 can be reduced to

$$[N - IC + M]\{x\} = \{y\}, \tag{7}$$

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with

$$\{x\} = \begin{bmatrix} \dfrac{\partial u}{\partial n}_1 \\[6pt] \dfrac{\partial u}{\partial n}_2 \\[6pt] \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial u}{\partial n}_{N-d} \\[6pt] un_{N-d+1} \\[4pt] \cdot \\ \cdot \\ \cdot \\ u_N \end{bmatrix}$$

Equation 7 represents a system of N equation for N unknowns. Upon solving for x Equation 3 can be used to find u(p) at any interior point. A four point Gauss-Legrendre numerical quadrature was successfully used to evaluate the off diagonal elements of A and M. This integration scheme was also used to find the solution at interior points.

The chief difficulty encountered when implementing this numerical scheme is integrating over the singularity contained in G (diagonal elements of A). In an effort to keep the boundary element code as general as possible, this integration is carried out numerically. The numerical procedure used follows the approach suggested by Hornbeck (Reference 3). The interval containing the singularity is broken into the two intervals shown in Figure 2; in this figure the singularity is located at X_1. A 24-point Gauss-Legendre numerical quadrature is used in the interval X_1 to $X_1 + .001$ and a four-point Gauss-Legendre quadrature is used from $X_1 + .001$ to X_2. Since the zeroes of the Legendre polynomials are clustered near the ends of the interval, subdividing the interval of interest places the maximum number of integration points near

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the singularity. This approach proved very accurate in evaluating the diagonal elements of A.

Figure 2. Division of the Integration Interval to Facilitate Accurate Intregration Around a Singularity

SYSTEMS GOVERNED BY POISSON'S EQUATION

1. THE FUNDAMENTAL SOLUTION

The general governing equation to steady heat conduction problems is the Poisson equation. The fundamental solution to this class of problems will be derived following the approach of Greenberg (Reference 4). As a result, the definition of a fundamental solution will be made clear.

The formal statement for this class of problem is

$$L(T) = \nabla^2 T - \phi = 0 \tag{8a}$$

where ϕ is a function of position, and B is the general two-dimensional region depicted in Figure 3. To complete the problem statement the mixed boundary conditions are specified,

$$C_1 T + C_2 \frac{\partial T}{\partial n} = \theta \text{ on } \Gamma \tag{8b}$$

Figure 3. Domain of Interest for the Poisson Equation

where, C_1 and C_2 are constants, and θ is a function of position. If G is defined as the fundamental solution to Equation 8a then,

$$\int_B G(\nabla^2 T - \phi)d\sigma = \int_B G\frac{\partial^2 T}{\partial x^2}d\sigma + \int_B G\frac{\partial^2 T}{\partial y^2}d\sigma - \int_B G\phi d\sigma \qquad (9)$$

Integrating the first term on the right-hand side yields

$$\int_B G\frac{\partial^2 T}{\partial x^2} dxdy = \int_{y_2}^{y_1} [G\frac{\partial T}{\partial x} - T\frac{\partial G}{\partial x}]_{x_1(y)}^{x_2(y)} dy + \int_B T\frac{\partial^2 G}{\partial x^2} d\sigma$$
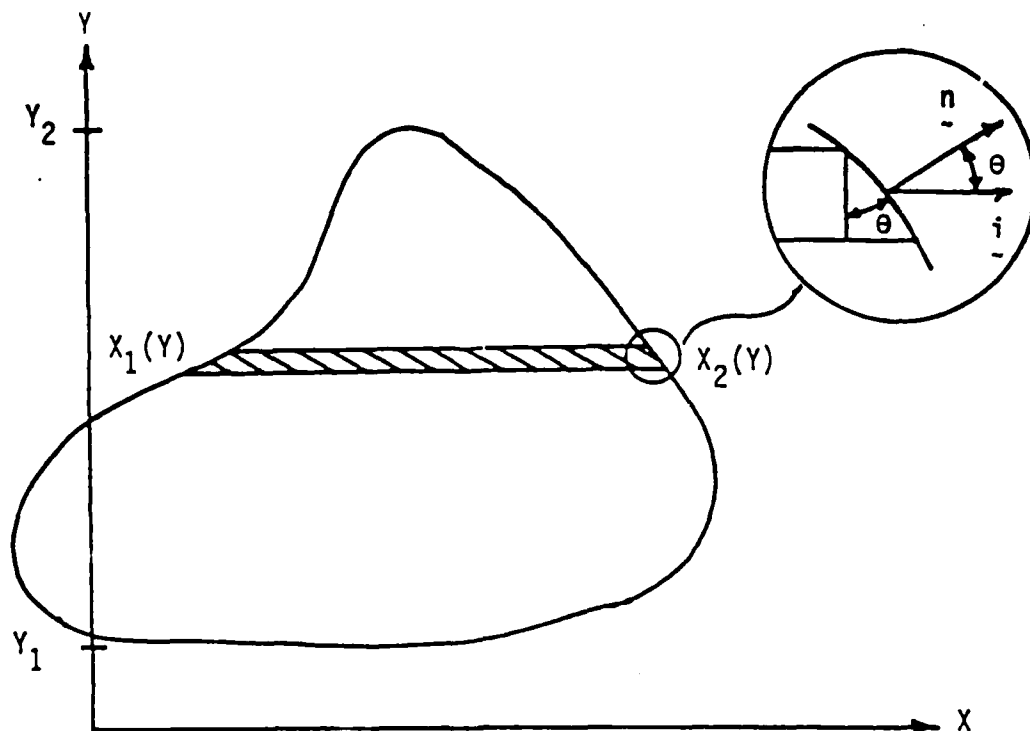
Referring to Figure 3, $dy\sin\beta d\ell = \underset{\sim}{n}\cdot\underset{\sim}{i}d\ell$; therefore,

$$\int_B G\frac{\partial^2 T}{\partial x^2} dxdy = \int_\Gamma [G\frac{\partial T}{\partial x} - T\frac{\partial G}{\partial x}] \underset{\sim}{i}\cdot\underset{\sim}{n}d\ell + \int_B T\frac{\partial^2 G}{\partial x^2}d\sigma \qquad (10)$$

similarly,

$$\int_B G\frac{\partial^2 T}{\partial y^2} d\sigma = \int_\Gamma [G\frac{\partial T}{\partial Y} - T\frac{\partial G}{\partial Y}]\underset{\sim}{j}\cdot\underset{\sim}{n}d\ell + \int_B T\frac{\partial^2 G}{\partial Y^2}d\sigma \qquad (11)$$

Substituting Equation 10 and 11 into Equation 9, noting that $\nabla^2 T - P = 0$, yields

$$0 = \int_\Gamma [G\frac{\partial T}{\partial n} - T\frac{\partial G}{\partial n}]d\ell + \int_B G\phi d\sigma + \int_B T\nabla^2 G d\sigma \qquad (12)$$

Now, if $\nabla^2 G = \delta(\underset{\sim}{r}-\underset{\sim}{r_0})$, where $\underset{\sim}{r_0}$ defines the position of any point in B, then

$$T(\underset{\sim}{r_0}) = \int_\Gamma [T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n}]d\ell - \int_B G\phi d\sigma$$

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This result is identical to Equation 3 for arbitrary mixed boundary conditions. In general the last integral in Equation 12 is of the form

$$\int_B TL*(G)d\sigma$$

where L* is the adjoint of L. In this case L is self adjoint, that is L = L*. The preceding paragraphs yield the conditions on G and the definition of the fundamental solution. Stated mathematically

$$\nabla^2 G = \delta(\underset{\sim}{r} - \underset{\sim}{r}_0)$$

Replacing the delta function by a delta sequence and writing the Laplace operator in polar coordinates, the conditions on G can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial G}{\partial r}) = k \lim_{\to \infty} \frac{ke^{-kr^2}}{\pi}$$

where
$$r = |\underset{\sim}{r}-\underset{\sim}{r}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

If $\frac{\partial G}{\partial r}(0)$ is finite, then,

$$G = \frac{1}{2\pi}\ln r + \lim_{k \to \infty}\frac{1}{2\pi}\int_r^1 \frac{e^{-kt^2}}{t}dt$$

Since the second term goes to zero in the limit, the fundamental solution to systems governed by Poisson's equation (which includes Laplace's equation) is given by

$$G = \frac{1}{2\pi}\ln r \qquad (13)$$

At this point a distinction between the fundamental solution and the Green's function should be made.

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$$GF = G + GB.$$

Here, GF is the Green's function to the governing equations and boundary conditions; G is the fundamental solution. GB is a function added to the fundamental solution to guarantee that the Green's function satisfies the boundary conditions necessary to make the first term in Equation 12 vanish. This term vanishing gives the direct solution to Equation 8a as

$$T(\underset{\sim}{r}_0) = \int_B GF \phi d\sigma$$

Greenberg (Reference 4) gives a more detailed discussion on this distinction.

2. IMPLEMENTATION OF THE FUNDAMENTAL SOLUTION

With the fundamental solution derived, the system defined by Equation 6 can be further specified. First the constant c, produced by the limit process of Equation 4, must be found. After substituting the fundamental solution into a condensed form of Equation 4, the limit looks like

$$\lim_{p \to P} u(p) = \frac{u(Q)}{2\pi} \lim_{p \to P} \int_\Gamma \frac{1}{r} d\ell - \frac{1}{2\pi} \frac{\partial u}{\partial n}(Q) \lim_{p \to P} \int_\Gamma r d\ell - \frac{1}{2\pi} \lim_{p \to P} \int \phi(p,Q) \ln r \, d\sigma \quad (14)$$

The evaluation of the first term on the right-hand side yields

$$\lim_{p \to P} \int \frac{1}{r} d\ell = \lim_{\epsilon \to 0} \left[\int_{\Gamma - \epsilon} \frac{1}{r} d\ell + \int_\epsilon \frac{1}{r} d\ell \right]$$

where ϵ is the radius of a small semicircular path circumventing the singularity on the boundary. In the limit

$$\lim_{p \to P} \int_\Gamma \frac{1}{r} d\ell = \int_\Gamma \frac{1}{r} d\ell + \pi;$$

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therefore, the contribution due to the singularity at $r = 0$ is π. The second integral in

$$\lim_{p \to P} \int_{\Gamma} \ln r \, d\ell = \lim_{\epsilon \to 0} \left[\int_{\Gamma - \epsilon} \ln r \, d\ell + \int \ln d\ell \right]$$

The second term on the left-hand side can be evaluated as

$$\lim_{\epsilon \to 0} \left[\epsilon \ln \epsilon - 1 \right] = 0;$$

hence, the contribution due to the singularity is zero. Since this is also true for the surface integral in Equation 14, its evaluation is omitted. The substitution of these integrals into Equation 4 reveals that $c = 1/2$.

The system of Equation 7 is now explicit, that is

$$A_{ji} = \frac{1}{2\pi} \int_{\Gamma_1} \frac{1}{r} \, d\ell_i ,$$

$$M_{ji} = \frac{1}{2\pi} \int_{\Gamma_2} \ln r \, d\ell_i ,$$

$$c = 1/2 \text{ and } r = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

The implementation of system 7 (Equation 7) is best described through the use of Figure 4. Each row in system 7 (Equation 7) is the result of placing the source point (x_j, y_j) at a node position on the boundary, then integrating its influence around Γ. The source point is then moved from node to node until the circuit is complete. The next two sections give examples of the implementation of this fundamental solution in the boundary element method.

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Figure 4. General Two-Dimensional Discreted Domain

3. STEADY HEAT CONDUCTION THROUGH AN INFINITE RECTANGULAR PRISM

The steady heat conduction in an infinite rectangular prism is governed by the two-dimensional Laplace's equation. The domain in this example is a square with a non-dimensional length of 6. Figure 5 gives the domain, the boundary conditions and the grid employed in the solution.

Figure 5. Discretized Square Domain With a Twelve-Point Grid
(Laplace Operator)

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Since the normal derivative is not well defined at the corners, nodes are not place in those locations. Jawson, et al.(Reference 5) discusses rounding the corners, Costable (Reference 6) discusses the use of Mellin transforms, and Liu and Sutton (Reference 7) discuss a double corner node method to negate the errors introduced by corners.

In this case, and all of the following cases, the material constants are unity. Table 1 presents numerical results for eight representative locations in the domain.

TABLE 1.   STEADY TEMPERATURE DISTRIBUTION IN A SQUARE

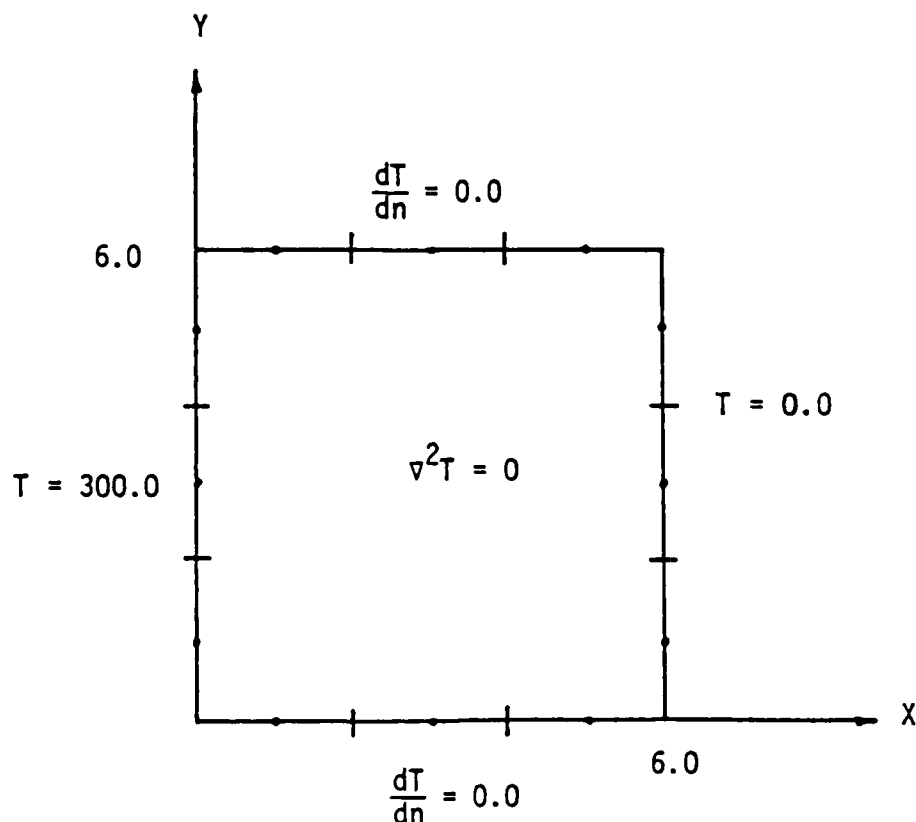| X	Y	B.E.M.	Analytical
2	2	200.54	200.0
2	3	200.40	200.0
2	4	200.54	200.0
2	5	200.79	200.0
3	2	150.01	150.0
3	3	150.01	150.0
3	4	150.01	150.0
3	5	149.96	150.0

4. STEADY HEAT CONDUCTION THROUGH AN INFINITE RECTANGULAR PRISM WITH
    A UNIT HEAT SOURCE

The addition of a unit heat source results in a Poisson governing equation. This section illustrates the solution to a simple steady non-homogeneous partial differential equation. In this case, the surface temperatures are zero. The same 12-point grid was also used in the solution to this problem. Table 2 gives numerical results at the 8 representative points.

TABLE 2. STEADY TEMPERATURE DISTRIBUTION IN A SQUARE WITH HEAT SOURCE

X	Y	B.E.M.	Analytical
2	2	2.175	2.172
2	3	2.399	2.397
2	4	2.175	2.172
2	5	1.442	1.436
3	2	2.399	2.397
3	3	2.654	2.652
3	4	2.399	2.397
3	5	1.520	1.521

SECTION IV

SYSTEMS GOVERNED BY THE NON-HOMOGENEOUS MODIFIED HELMHOLTZ EQUATION

1. THE FUNDAMENTAL SOLUTION

The remaining five systems, either directly or via the Laplace transform, are governed by the non-homogeneous modified Helmholtz equation. The general problem statement is as follows:

$$\nabla^2 T - k^2 T = \phi_1 \tag{15a}$$

with boundary conditions

$$C_1 T + C_2 \frac{\partial T}{\partial n} = \theta \tag{15b}$$

where C_1 and C_2 are constants and θ is a function of position. Following the definition of the fundamental solution given in Section 3, and noting that the modified Helmholtz operator is self adjoint, gives the equation for the fundamental solutions as

$$\nabla^2 G - k^2 G = \delta(x - x_0, y - y_0) \tag{16}$$

If the Laplacian is written in polar coordinates, the stretching $\rho = kr$ is introduced and G is restricted to a function of r only, Equation 16 reduces to a zero order modified Bessel equation.

$$\rho^2 \frac{\partial^2 G}{\partial \rho^2} + \rho \frac{\partial G}{\partial \rho} - \rho^2 G = \delta(\rho - \rho_0) \tag{17}$$

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Since G should be singular at $r = 0$, the desired solution to Equation 17 is the zero order modified Bessel function of the second kind.

$$G = C_3 K_0(kr)$$

The constant C_3 is a measure of the strength of the singularity at $r = 0$. It can be evaluated by substituting G into Equation 17 and integrating over an arbitrarily small disk encompassing the singularity.

$$C_3 \lim_{\varepsilon \to 0} \int_B \nabla^2 K_0(k\varepsilon)d\sigma - C_3 \lim_{\varepsilon \to 0} \int_B k^2 K_0(k\varepsilon)d\sigma = \lim_{\varepsilon \to 0} \int_B \delta(\varepsilon)d\sigma$$

By definition, the right-hand side is unity. By noting $K_0(kr) = - \ln(kr)$ for $|r| < \varepsilon$, the second integral on the left-hand side evaluates to $- C_3 \pi k^3 [(kr)^2 \ln(kr) - 1/2(kr)^2]$, and in the limit goes to zero. Since G is a function of r only, $\frac{\partial G}{\partial r}$ is the only non zero normal derivative.

$$C_3 \lim_{\varepsilon \to 0} \int_B \nabla^2 K_0(k\varepsilon)d\sigma = - \lim_{\varepsilon \to 0} \int C_3 k K_1(k\varepsilon)d\ell$$

For $|r| < \varepsilon$, $K_1(kr) = 1/kr$; therefore, this integral evaluates to $-C_3 2\pi$. Hence, $C_3 = 1/2$ and the fundamental solution to the general modified Helmholtz equation is

$$G = \frac{-1}{2\pi} K_0(kr) \tag{18}$$

Before proceeding with the numerical procedure outlined in Section II, the constant c of Equation 4 must be determined. Since this process is identical to that in Section III, the details are omitted. For piecewise smooth surfaces and $G = -1/2 \pi k_0(kr)$, $c = 3/2$. Having found the fundamental solution and c,

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boundary element methods can now be used in the solution to a more complex class of heat conduction problems.

As an aside, the fundamental solution in Equation 18 differs from that of Rizzo and Shippy (Reference 8) by a negative sign. In their case, $c = 1/2$. The reason for the difference is in their approach to finding the fundamental solution. They follow Boley and Weiner (Reference 9) in that Rizzo and Shippy find the solution to the conduction equation when subjected to a delta function heat source. They then Laplace transform the conduction equation to get the modified Helmholtz equation and Laplace transform the solution to get the fundamental solution. Both fundamental solutions give identical temperature distributions.

2. STEADY HEAT CONDUCTION THROUGH A THIN RECTANGULAR PLATE

Steady heat conduction through a thin plate is the most elementary case in the Helmholtz operator class. It is steady, homogeneous and the boundary conditions are constant.

$$\nabla^2 T - T = 0,$$

$$T = 1 \text{ on } r.$$

A twelve-point grid is used in the solution to this problem. Figure 6 shows the domain and boundary conditions and, again, all material constants are unity. The results are given in Table 3.

Figure 6.   Discretized Square Domain With a Twelve-Point Grid
(Helmholtz Operator)

TABLE 3.   STEADY TEMPERATURE DISTRIBUTION IN A THIN RECTANGULAR PLATE

X	Y	B.E.M.	Analytical
.2	.2	.4454	.4439
.2	.3	.2928	.2992
.2	.4	.1991	.2056
.2	.5	.1382	.1423
.3	.2	.5499	.5404
.3	.3	.3869	.3853
.3	.4	.2717	.2731
.3	.5	.1902	.1921

3. TRANSIENT HEAT CONDUCTION THROUGH AN INFINITE PRISM OF CONSTANT CROSS SECTION

The problems discussed hereafter fall under the modified Helmoltz operator class via the Laplace transform. In this section, the Laplace transform approach to boundary element methods is introduced. First, the heat conduction through circular and rectangular prisms (both with unit temperature on the boundary) is discussed. Then, the problem of constant heat production in an infinite cylinder is discussed. Finally, the problem of heat conduction from an infinite circular prism with heat radiated from its surface is discussed. All the numerical solutions are compared to the analytic solutions given by Carslaw and Jaeger (Reference 10).

Before proceeding with numerical examples, the corresponding transform domain system must be developed. The general heat conduction equation is stated as

$$\nabla^2 T = \frac{\partial T}{\partial t} \tag{19a}$$

and

$$C_1 T + C_2 \frac{\partial T}{\partial n} = \theta \tag{19b}$$

where C_1 and C_2 are constants, and is a function of position and time. Following Rizzo and Shippy (Reference 8), the Laplace transform of Equation 19a and boundary conditions Equation 19b yields

$$k\nabla^2 \bar{T} - s\bar{T} = 0 \tag{20a}$$

and

$$C_1 \bar{T} + C_2 \frac{\partial \bar{T}}{\partial n} = \bar{\theta} \tag{20b}$$

where $\bar{T}$ signifies the Laplace transform of T and s is the Laplace transform

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parameter. In Equation 20, it is assumed $T(0,r) = 0$. Notice Equation 20a
is identical to Equation 15 with $\phi_1 = 0$ and $k^2 = s/k$. The frequency domain
system can then be solved by using the fundamental solution defined by equation
18. The time domain solution can be retrieved through the inverse Laplace
transform. In each of the following cases, the least square method of Schapery
(Reference 11) is used to perform the inverse Laplace transform. Rizzo and Shippy
(Reference 8) also give a complete discussion of this method. In their paper, Rizzo
and Shippy use conditions on the frequency response for large frequency and on the
time response for small time to find the steady state constants. The accuracy
of this method seems to be better when the steady state constants are found
numerically.

4. HEAT CONDUCTION THROUGH INFINITE PRISMS OF UNIFORM CROSS SECTION WITH UNIT SURFACE TEMPERATURE

The solution to transient heat conduction through an infinite square prism
is given. In the time domain the surface temperature is unity, this transforms
to 1/S. The frequency domain problem statement is

$$\nabla^2 T - sT = 0 \tag{21a}$$

and

$$T = 1/s \text{ on } \Gamma. \tag{21b}$$

An evenly spaced twenty point grid is used in the solution of this problem.
Figure 7 gives the time responses found with Equation 20.

Next, the solution to heat conduction through an infinite circular prism
with unit surface temperature is presented. Equations 21 also given this
problem, in this case however, an evenly spaced 24-point grid was used.
Figure 8 presents the time response for a representative point in the domain.
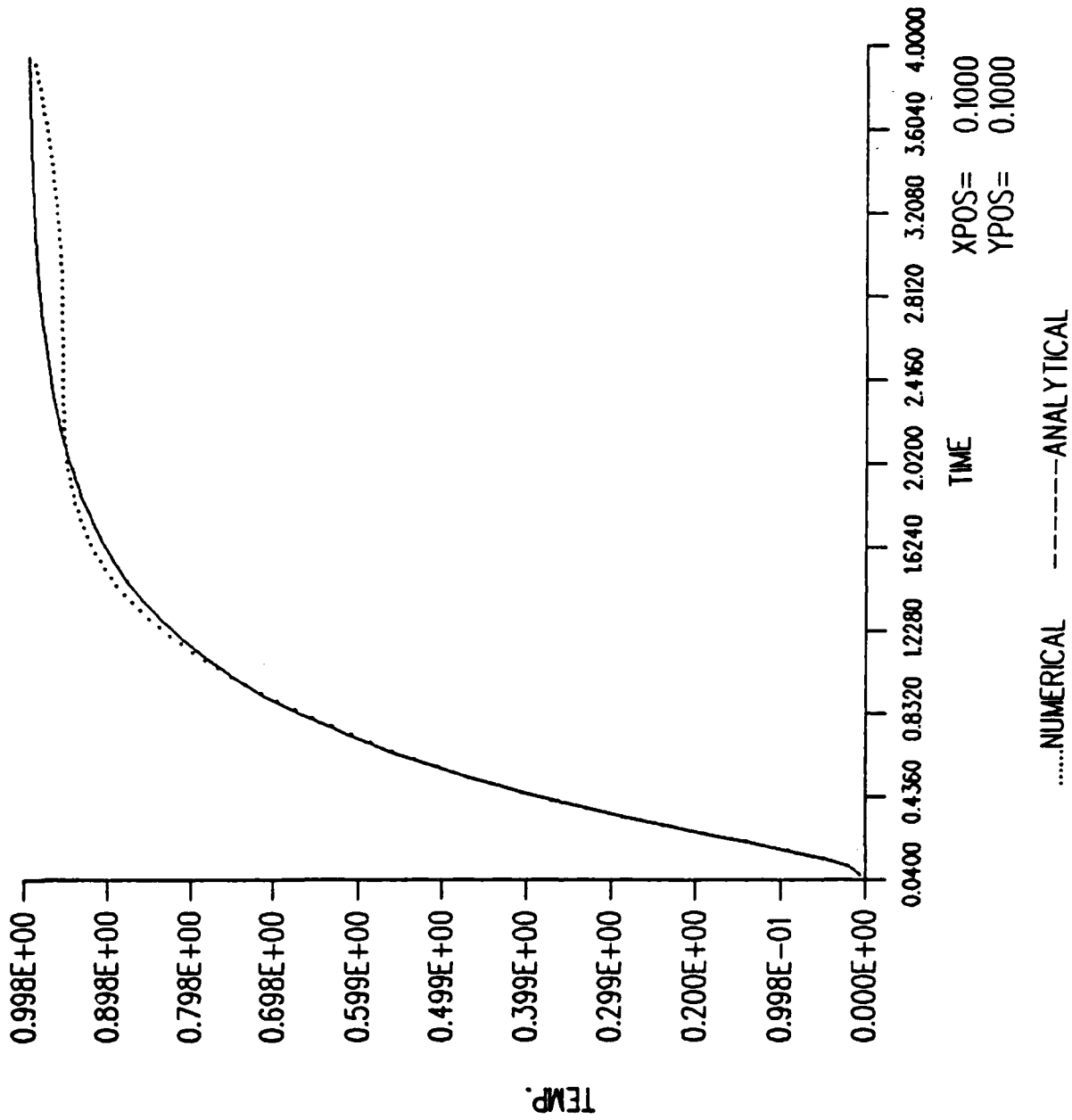The accuracy for all other points is comparable.

Figure 7. Time Response of the Temperature in a Square With Uniform Temperature on the Surface
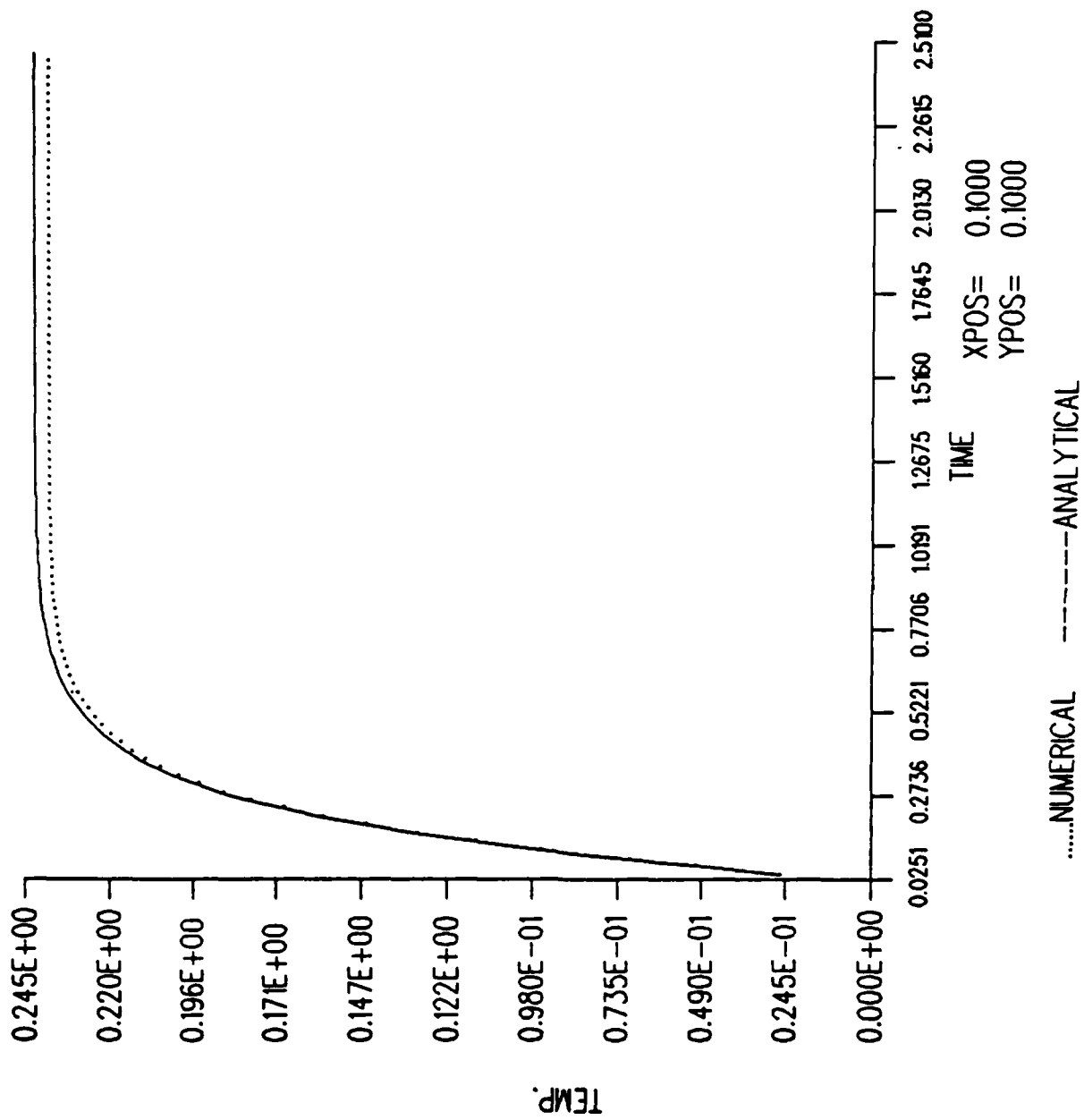
XPOS= 0.1000
YPOS= 0.1000

TIME

...NUMERICAL   ----ANALYTICAL

TEMP.

21

Figure 8. Time Response of the Temperature in a Circle With Uniform
Temperature on the Surface

5. HEAT CONDUCTION THROUGH AN INFINITE CIRCULAR CYLINDER WITH CONSTANT HEAT PRODUCTION

The next step in complexity is produced by the addition of constant heat production. Here, the initial and surface temperatures are zero; heat is produced at a unit rate per unit volume. The frequency domain governing equation is

$$\nabla^2 T - sT = -1$$

with

$$T = 0 \quad \text{on} \quad r = 1$$

This example shows the B.E.M. can accurately give the solution to frequency domain, non-homogeneous modified Helmholtz equation and hence also give the time response. The same domain and 24-point grid is used in this solution. Figure 9 gives the time response.

6. HEAT CONDUCTION THROUGH AN INFINITE CIRCULAR CYLINDER WITH RADIATION AT THE SURFACE

This final example discusses the heat conduction produced in a cylinder when heat is radiated from its surface. This example is different in that functional values for the boundary conditions are not explicit, rather a relationship between the temperature and its normal derivative is given.

$$\nabla^2 T - sT = 0 \tag{22a}$$

and

$$\frac{\partial T}{\partial n} + T = \frac{1}{s} \tag{22b}$$

Substituting boundary conditions Equation 22b into integral Equation 4 gives a boundary element equation in the form of

$$\frac{1}{2}T = T \int_{\Gamma} \left(\frac{\partial G}{\partial n} + G \right) d\ell - \frac{1}{s} \int_{\Gamma} G d\ell$$
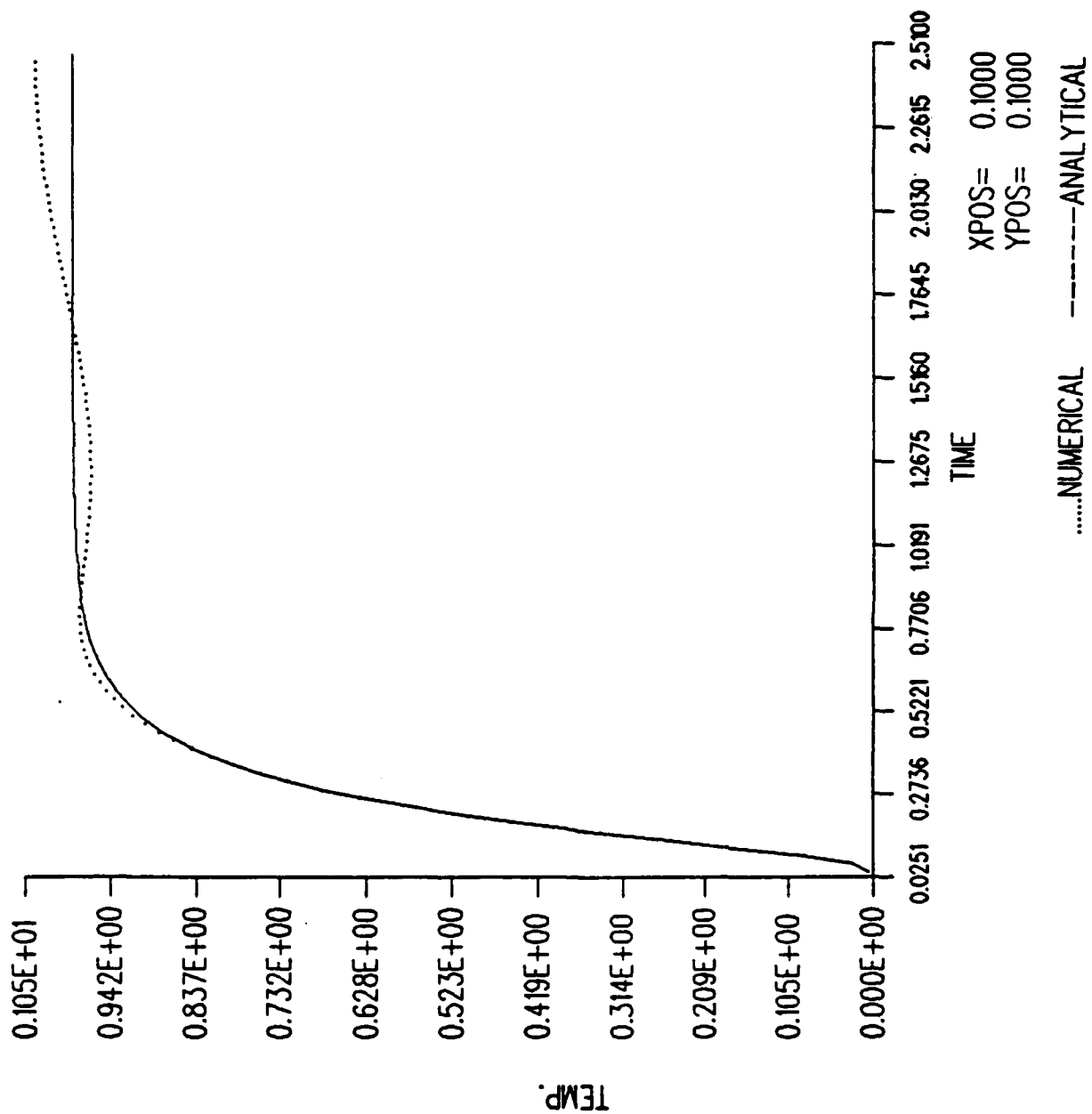
23

Figure 9.   Time Response of the Temperature in a Circle With Constant Heat Production

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Equivalently, the fundamental solution, its normal derivative and the boundary conditions can be redefined as

$$\frac{\partial G^*}{\partial n} \equiv \frac{\partial G}{\partial n} + G , \quad G^* = G$$

and

$$\frac{\partial T}{\partial n} = \frac{1}{s} \quad \text{on } \Gamma$$

where G is the fundamental solution to the modified Helmholtz equation. This reformulation precipitates the use of the boundary element method. Again, the same domain and grid were used in the solution to this problem. The time response is given in Figure 10.
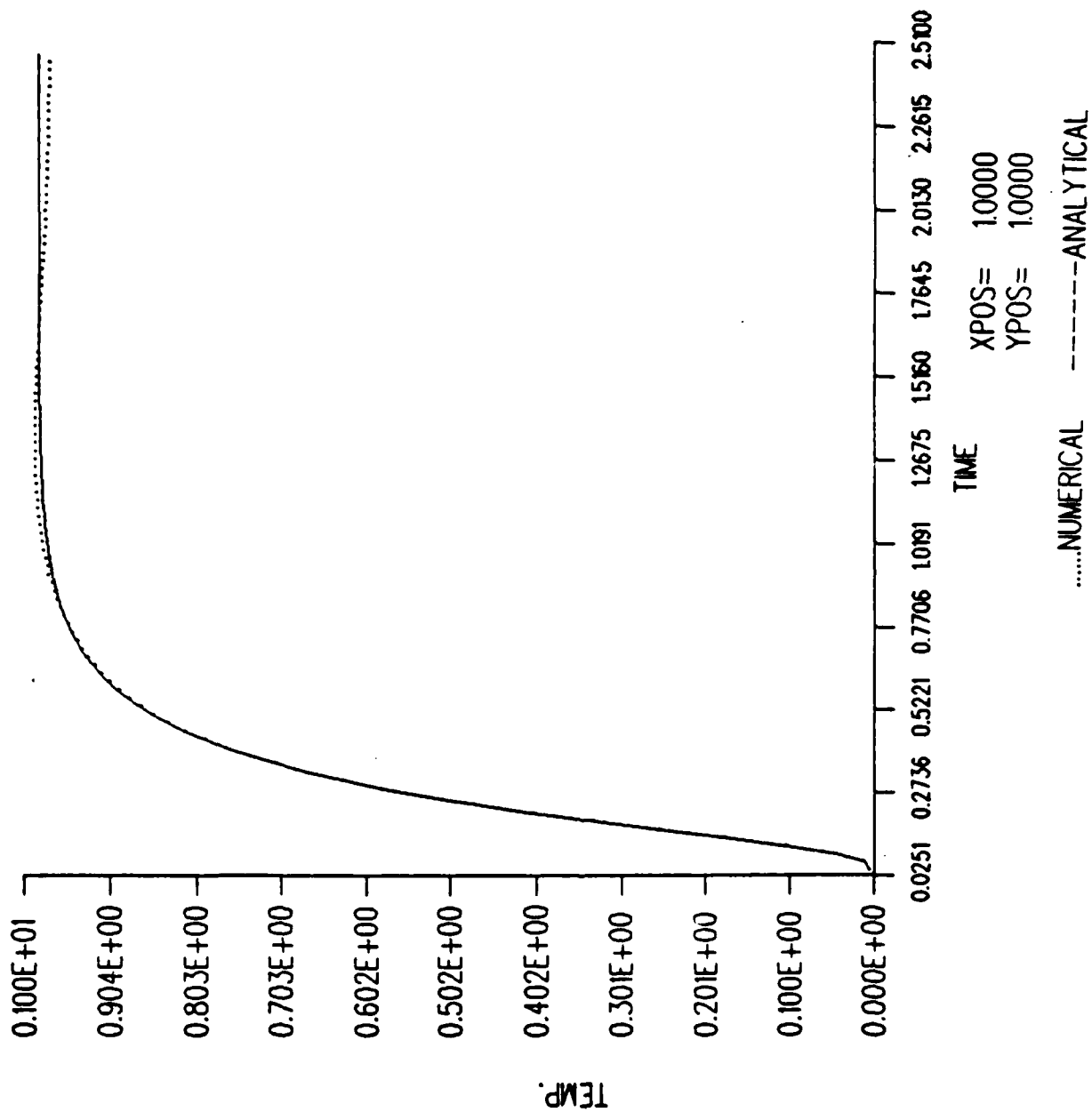
Figure 10. Time Response of the Temperature in a Circle With Radiation at its Surface

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SECTION V

DISCUSSION

As seen in the previous examples, the accuracy of the constant element-boundary element method is excellent for both steady and transient problems. However, this approach gives poor and sometimes spurious results for complex geometries. As shown by Cruse (Reference 12), the accuracy of the B.E.M. can be dramatically increased by employing linear elements. For simple goemetries the small improvement in accuracy produced by linear elements does not justify the increase in computer time.

In general, for a given accuracy level the B.E.M. solutions are much more efficient than finite difference or finite elements solutions (Reference 13); however, the surface integral introduced by non-homogeneous governing equations can greatly increase the run time of a boundary integral solution. To combat this problem, it is important to take advantage of symmetry when ever possible. The run time of the example presented in subsection IV.5 was reduced up to 85% when symmetry considerations were invoked.

The chief purpose of this report is to explore the boundary element solution to transient problems. Hence, it is important to note that the time dependent solutions consistently breakdown for large time. This is exemplified by the solution to heat conducting through an infinite square prism for large time (Figure 11). Rizzo and Cruse (Reference 14) point out that this behavior is attributable to the inverse Laplace transform. Bellman, et al. (Reference 15) shows that numerical inverse Laplace transforms inherently grow unstable for large time. For short duration events, such as elastic or plastic wave propagation, the instability does not present a problem (Reference 16). In the case of long duration events, such as creep,
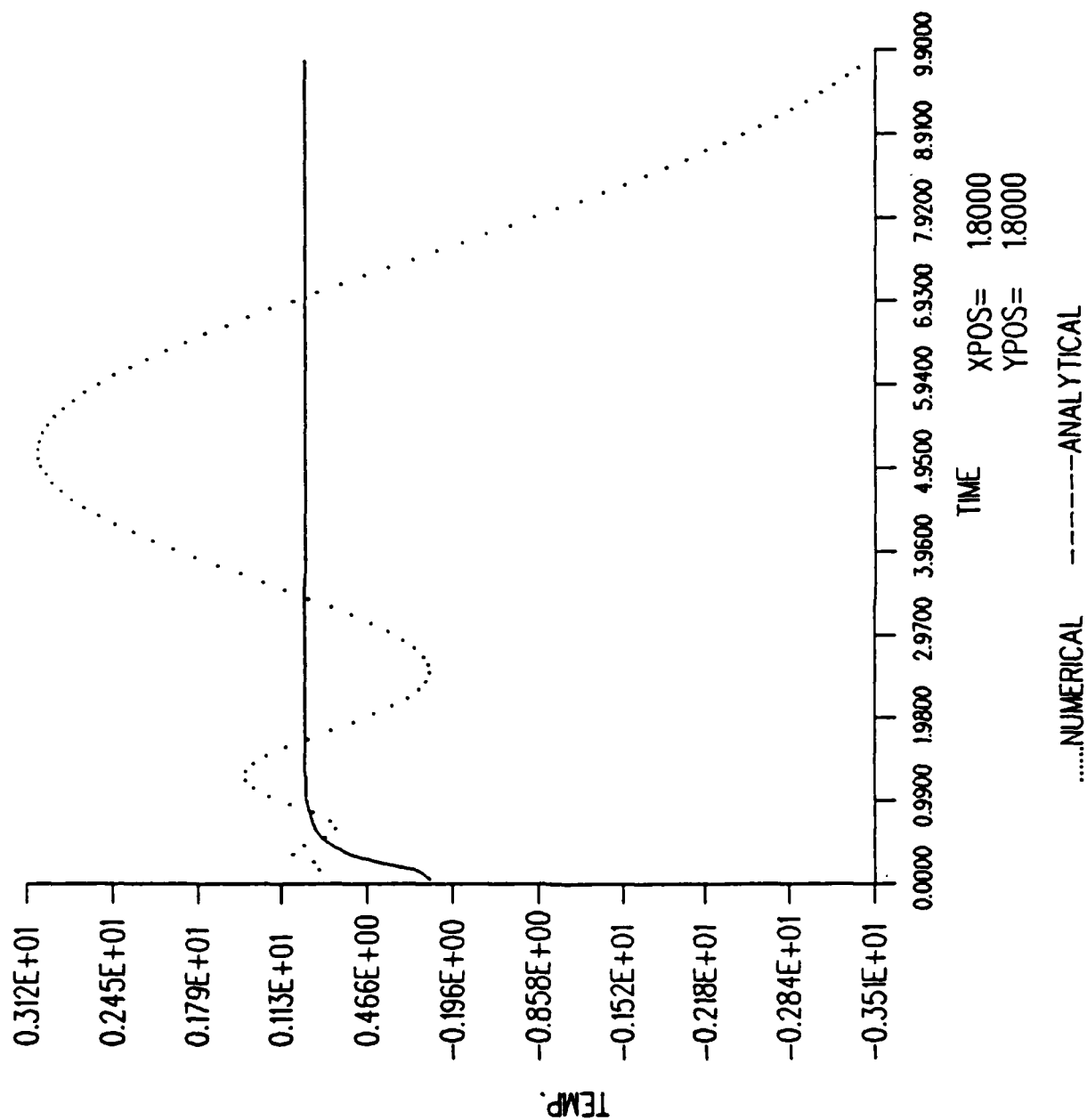
Figure 11. Time Response of the Temperature in a Circle With Unit Temperature on its Surface (For Large Time)

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a time dependent fundamental solution can be derived (Reference 17). This

approach is not as desirable as it might seem since the B.E.M. solution

must be evaluated at each time step, hence, greatly increasing computer time.

The building block fashion in which this report is presented is intended

to aid in the development of transient boundary element codes by providing

increasingly more difficult test cases. To this purpose, it is suggested

that the two-dimensional codes given by either Crouch and Starfield (Reference

18) or Brebbia et al. (Reference 19) be used as a starting point. Both these

codes can be expanded to accommodate non-homogeneous, time dependent problems.

SECTION VI

CONCLUSIONS

The success of the boundary element method in the solution of two-dimensional transient heat conduction problems has been reaffirmed. A method to find the fundamental solution is presented: also a distribution between the fundamental solution and Green's function is discussed. The important numerical considerations lie in the numerical integration around the singularity of the fundamental solution and in the instability of the inverse Laplace transform for large time. The accuracy and efficiency of the boundary element method in the solution of linear transient governing equations represent a significant improvement over other available numerical schemes.

REFERENCES

1. Rizzo, F. J., The Boundary-Integral Equation Method: A Modern Computational Procedure in Applied Mechanics , Boundary-Integral Equation Method: Computational Applications in Applied Mechanics, ed. T. A. Cruse and F. J. Rizzo, ASME, N.Y., 1975.

2. Brebbia, C. A., The Boundary Element Method for Engineers, John Wiley and Sons, Inc., N.Y., 1978.

3. Hornbeck, R. W., Numerical Methods, Prentice-Hall, Inc., N.J., 1975.

4. Greenberg, M. D., Applications of Green's Functions in Science and Engineering, Prentice-Hall, Inc., N.J., 1975.

5. Jawson, M. A., Matti, M. and Symm, G. T., "Numerical Biharmonic Analysis and Some Applications," Int. J. Solids Struc., Vol. 3, pp. 309-332, 1967.

6. Costable, M. and Stephan, E. B., "The Method of Mellin Transforms for Boundary Integral Equations on Curves with Corners," Numerical Solution of Singular Integral Equations, ed. A. Gerasoulis and R. Vichnevetsky, p. 95, 1984.

7. Liu, J. C. and Sutton, M. A., "General Two-Dimensional Transient Heat Conduction Problems Using Boundary Integral Equation Method with New Corner Approximation," SECTAM XIII Proceedings, p. 603, 1986.

8. Rizzo, F. J. and Shippy, D. J. "A Method of Solution for Certain Problems of Transient Heat Conduction," AIAA Jou., Vol. 8, No. 11, pp. 2004-2009, 1970.

9. Boley, B. A. and Weiner, J. H., The Theory of Thermal Stresses, John Wiley and Sons, Inc., N.Y., 1960.

10. Carslaw,. S. and Jaeger, J. C., Conduction of Heat in Solids, Oxford Press, London, 1959.

11. Schapery, R. A., "Approximate Methods of Transform Inversion for Viscoelastic Stress Analysis," Proceedings of the Fourth U. S. National Congress on Applied Mechanics, Vol. 2, pp. 1075-1085, 1962.

12. Cruse, T. A., "Two-Dimensional BIE Fracture Mechanics Analysis," Appl. Math. Modelling, Vol. 2, pp. 287-293, 1978.

13. Fenner, R. T., "The Boundary Integral Equation (Boundary Element) Method in Engineering Stress Analysis," The Boundary Integral Equation Method in Stress Analysis, Mechanical Engineering Publications Ltd., London, 1983.

14. Rizzo, F. J. and Cruse, T. A., "A Direct Formulation and Numerical Solution of the General Transient Elastodynamic Problem I," Jou. of Math. Anal. and Appl., Vol. 22, pp. 244-259, 1968.

15. Bellman, R. E., Kalaba, R. E. and Lockett, J., Numerical Inversion of the Laplace Transform, American Elsevier, N.Y., 1966.

16. Cruse, T. A., "Direct Formulation and Numerical Solution of the General Transient Elastodynamic Problem II," Jou. of Mau. Anal. and Appl., Vol. 22, pp. 341-355, 1968.

17. Brebbia, C., Progress in Boundary Element Methods, John Wiley and Sons, N.Y., 1981.

18. Crouch, S. L. and Starfield, A. M., Boundary Element Methods in Solid Mechanics (With Applications to Rock Mechanics and Geological Engineering), George Allen and Unwin Ltd., London, 1983.

19. Brebbia, C. A., Telles, J. C. F. and Wrobel, L. C., Boundary Element Techniques: Theory and Applications, Springer-Verlag, N.Y., 1984.

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